Quantumlike corrections and semiclassical description of charged-particle beam transport

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(Received 6 October 1997)

It is shown that the standard classical picture of charged-particle beam transport in paraxial approximation may be conveniently replaced by a Wigner-like picture in a *semiclassical approximation*. In this *effective* description, the classical phase-space equation for electronic rays is replaced by a *von Neumann–like equation*, where the transverse emittance plays the role of $\hbar$. Relevant remarks concerning the quantumlike corrections for an arbitrary potential in comparison with the standard classical description of the beam transport are given. [S1063-651X(98)07506-0]

PACS number(s): 41.85.−p, 03.65.Ca, 42.50.−p

I. INTRODUCTION

Quantum formalism for describing a number of macroscopic systems, such as plasmas, linear and nonlinear electromagnetic (em) radiation beam propagation (for instance, optical fibers, transmission lines), em traps, charged-particle beam transport, etc., have received a great deal of attention during the last two decades [1]. For these nonproper quantum systems, it is appropriate to say *quantumlike* description instead of the proper quantum one, because the physics involved, which is basically classical, can be described by formally replacing the Planck's constant with a suitable fundamental parameter of the particular system considered. A quantumlike theory of light rays was, for example, constructed by Gloge and Marcuse [2] in order to recover wave optics starting from a formal quantization of geometrical optics based on Fermat's principle. In particular, this procedure has allowed one to recover, in paraxial approximation, the Schrödinger-like equation for the em field, the so-called Fock-Leontovich equation [3], widely used in linear and nonlinear em radiation optics [4–6]. The *transition* from geometrical optics (the analogous of classical mechanics) to wave optics (the analogous of wave mechanics) was performed by introducing some correspondence rules, fully similar to the Bohr's ones, in which $\hbar$ is replaced by $\lambda/2\pi$, the inverse of the wave number ($\lambda/2\pi\equiv 1/k$). In particular, in this context, the paraxial approximation (the analog of the nonrelativistic approximation of quantum mechanics) describes the radiation beam transport in an arbitrary medium and the corresponding quantumlike formalism (a quantumlike uncertainty principle included) and Fock-Leontovich equation can be fully recovered by formally replacing $\hbar$ with $\lambda/2\pi$ in the nonrelativistic quantum mechanics [2]. This fruitful procedure has been provided for transferring algorithms and many solutions of quantum mechanics to radiation beam physics, especially for optical fibers [7,8], coherent and squeezed states theories [9–14], Schrödinger cat states [15,16], and phase-space investigations within a Wigner-like picture [17] in which a quasiclassical distribution, fully similar to quantum Wigner transform [18] governs the paraxial em ray evolution.

More recently, a procedure similar to that of *Gloge and Marcuse* has allowed one to construct a quantumlike model of charged-particle beam transport in both real space and phase space, called *thermal wave model* (TWM) [19]. This model has been applied to a number of problems of charged-particle beam optics and dynamics [20–25]. It assumes that the particle beam evolution is governed by a Schrödinger-like equation for a complex function, the so-called beam wave function (BWF) whose squared modulus is proportional to the beam density where Planck's constant is replaced by the *beam emittance* [26]. In particular, in TWM framework, a Wigner-like transform, seems to be useful and appropriate to give the quantum-like phase-space description of particle beams [25].

In this paper, we want to suggest an approach alternative to the one that is similar to the *Gloge and Marcuse* procedure given by TWM. By starting from the electronic ray concept given in electron optics, we review the standard electronic ray approach to charged-particle beam optics and dynamics and introduce an effective description of the beam transport, which takes into account the thermal spreading among the electronic rays. In the following sections, we start from the electronic ray concept and introduce the paraxial-ray approximation. In Sec. II, the paraxial-ray equation is solved for the case of a linear lens (Hill's equation), while in Sec. III the statistical description of electronic rays allows us to obtain some important results such as the virial description of the beam and a quantumlike uncertainty relation. A two-dimensional (2D) phase-space description of the electronic rays is performed in Sec. IV, where, in the paraxial approximation, we show that an *effective* description can be given in terms of a *quasidistribution* in the phase space, which plays a role analogous to the one played in quantum mechanics by the Wigner function for pure states [18]. An analysis of the *quantumlike corrections* that the above effective approach gives is presented in Sec. V where a comparison with the classical approach up to the 4th-order moment description of the system for an arbitrary potential is performed. It is shown that, for dilute and paraxial beams, the discrepancies are neg-

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ligible. Finally, in Sec. VI we summarize the conclusions and give some remarks that are relevant for charged-particle and em beam transport as well as for quantum optics and very recent investigations in constructing positive definite distribution functions such as the one used in symplectic tomography [27–29].

A. The concept of electronic rays and the paraxial electronic-ray approximation

It is well known that *electron optics* [30] has been developed by using the similarity between charged-particle motion and the behavior of the light rays in geometrical optics. For nonrelativistic particle motion, this analogy shows that potential energy and particle trajectories play the role fully similar to the ones played by refractive index and light rays, respectively. In particular, this similarity allows us to introduce the concept of *electronic rays*. On the basis of this optical language, refraction and reflection laws for electronic rays can be introduced and their formulation is fully similar to the one that is used for light rays. The basic electron optics concepts have been developed in connection with the first experimental investigations of charged-particle motion (ions and electrons) in oscilloscopes and mass spectrometers. However, electron optics have been rapidly developed and applied to electron microscopy [31], electro-optical transducers [32], particle accelerators [33,34], etc.

When the potential is a function of the coordinates, it corresponds to an inhomogenous refractive index, and the electron trajectory through this inhomogenous potential region corresponds to a light ray through an inhomogeneous medium. In case we have several particles moving together in an arbitrary potential, each particle trajectory is an electronic ray.

In order to consider a charged particle beam as a special case of the above particle system, we introduce the so-called *paraxial electronic ray approximation* [33]. In this case, the system has a special direction, the instantaneous propagation direction, say z, and the following conditions hold:

$$\dot{x} \equiv \frac{dx}{dz} \ll 1, \quad \dot{y} \equiv \frac{dy}{dz} \ll 1, \tag{1}$$

where x and y are the transverse (with respect to z) coordinates. In other words, paraxial approximation corresponds to a very small deviation of the electronic rays from the propagation direction. Note that, in principle, the beam particles may have a relativistic motion along z (longitudinal motion) but, in order to be consistent with the paraxial approximation, their transverse motion must be nonrelativistic ($v_\perp \equiv \sqrt{v_x^2+v_y^2} \ll c$).

Let us consider a beam so dilute that the space charge effects can be considered negligible. If the thermal spreading of the particle velocity is negligible, in the case of aberrationless focusing, the particle converges in one point F only (focal point). Of course, if the thermal spreading is taken into account, the above circumstance will be modified. In fact, the beam will not focus at only one point and, if the electron rays are initially parallel, they will diverge and the beam naturally defocuses.

In order to go deep into the *thermal spreading among the electronic rays*, in the next section we consider the single particle motion in a linear lens and in the section later a statistical treatment of the electronic rays will be performed.

II. SINGLE-PARTICLE MOTION (SINGLE ELECTRONIC RAY)

Let us consider for simplicity the particle motion in the 2D case: for instance, the y component of the particle motion is neglected. Typically, the Hamiltonian for the x component motion of a single charged-particle with rest mass m_0 is given in the following dimensionless form:

$$H = \frac{p^2}{2} + U(x,z), \tag{2}$$

where $p = \dot{x}$ is the canonical conjugate momentum. Note that Eq. (2) describes a 1D motion (along x) of a classical particle when z plays the role of a timelike variable and U is an effective dimensionless potential energy, which can be expressed in terms of a polynomial form in x of arbitrary degree N as

$$U(x,z) = \frac{k_0(z)}{1!}x + \frac{k_1(z)}{2!}x^2 + \frac{k_2(z)}{3!}x^3 + \frac{k_3(z)}{4!}x^4 + \cdots$$
$$= \Sigma_{n=0}^{N} \frac{k_n}{(n+1)!} x^{n+1}. \tag{3}$$

U has been made dimensionless, dividing the effective energy potential of the system by the relativistic longitudinal energy $m_0\gamma_0 c^2 \equiv mc^2$ (γ_0 being the longitudinal relativistic factor). In particular, for a pure quadrupolelike potential (linear lens) Eq. (2) becomes

$$H = \frac{p^2}{2} + \frac{k_1(z)}{2}x^2. \tag{4}$$

Let us consider the equation of motion that follows from Eq. (4) (the Hill's equation [33,35]):

$$\ddot{x} + k_1(z)x = 0, \tag{5}$$

where $\dot{p} = -k_1(z)x$. The general solution of Eq. (5) can be put in the following form

$$x = \sqrt{2}E(z)\cos[\phi(z) - \phi_0] \equiv \sqrt{2}E(z)\cos\Delta\phi(z), \tag{6}$$

where ϕ_0 is an arbitrary constant and $E(z)$ is a function defined unless an arbitrary constant factor. By imposing that Eq. (6) is a solution of Eq. (5), we easily obtain the following conditions:

$$E^2\Delta\dot{\phi} = \text{const} \equiv \frac{I_0}{2}, \tag{7}$$

and

$$\ddot{E} + k_1 E - \frac{I_0^2}{4E^3} = 0. \tag{8}$$

Moreover, it is easy to prove that x and p satisfy the following quadratic form:

$$\mathcal{J}(x,p,z)=\gamma(z)x^2+2\alpha(z)xp+\beta(z)p^2=\frac{I_0}{2}, \tag{9}$$

where

$$\gamma(z)=\frac{I_0}{4E^2}+\frac{\dot{E}^2}{I_0}, \quad \alpha(z)=-\frac{E\dot{E}}{I_0}=-\frac{1}{2I_0}\frac{dE^2}{dz},$$

$$\beta(z)=\frac{E^2}{I_0}, \tag{10}$$

are called *Twiss parameters* [35]. Note that $\mathcal{J}(x,p,z)$ is an invariant for the Hamiltonian (4), namely,

$$\frac{\partial \mathcal{J}}{\partial z}+\{\mathcal{J},H\}=0, \tag{11}$$

where $\{\cdots\}$ denotes the classical Poisson brackets [36]. It is worth noting that the invariant that is quadratic form in coordinates and momentum for the parametric classical oscillator is known as the Ermakov invariant [37] and its quantum analog was found by Lewis [38] and discussed in [39]. It is easy to see that the determinant of the matrix associated with the quadratic form (9) is conserved:

$$\gamma\beta-\alpha^2=\tfrac{1}{4}. \tag{12}$$

Thus, from Eqs. (10)–(12) we obtain the identity

$$\frac{I_0^2}{4}=\left(\dot{E}^2+\frac{I_0^2}{4E^2}\right)E^2-(\dot{E}E)^2, \tag{13}$$

and the following inequality, which will be used later:

$$E\left(\dot{E}^2+\frac{I_0^2}{4E^2}\right)^{1/2}\geqslant\frac{I_0}{2}. \tag{14}$$

III. STATISTICAL DESCRIPTION OF ELECTRONIC RAYS

The results of the previous section can be used now to describe statistically the spreading among the electronic rays in a linear lens.

First of all, we observe that solution (6) is typically considered in particle accelerators for the case of a very smooth $k_1(z)$ compared to the variation of the *phase advance* $\Delta\phi(z)$ [33,40]. Also the amplitude $E(z)$ is typically a very slow function compared to $\Delta\phi(z)$ [33,40]. It is easy to see that in these circumstances the paraxial approximation is naturally satisfied. In fact, for an arbitrary initial transverse-space particle distribution, most of the particle trajectories remain confined in a limited region (if suitable stability conditions hold). Consequently, in the statistical description it can be assumed that this region represents a sort of *mean spread* for the generic particle position or, equivalently, a *mean spot* for a generic electronic ray corresponding to the most probable phase-space accessible region. This way, we can introduce also the average of an arbitrary observable. In particular, in order to estimate the above spot size we have to compute the following rms definition:

$$\sigma_x{}^2\equiv\langle x^2\rangle=\equiv\lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t)dt. \tag{15}$$

Since $x(z)$ contains a fast-period dependence on z, one can replace Eq. (15) with an average on the phase

$$\sigma_x^2=\langle x^2\rangle=\frac{1}{2\pi}\int_0^{2\pi}x^2 d\Delta\phi, \tag{16}$$

which gives [the average is performed only on the fast time scale, where $E(z)$ is almost constant]

$$\sigma_x^2(z)=\langle x^2\rangle=E^2(z). \tag{17}$$

Consequently, the instantaneous amplitude of solution (6) of the electronic ray equation in a linear lens corresponds to the statistical estimate of the transverse beam spot size σ_x. Similarly, we define the rms of the electronic ray slope $p\equiv dx/dz=\sqrt{2}\dot{E}\cos\Delta\phi-(I_0/\sqrt{2}E)\sin\Delta\phi$, obtaining

$$\sigma_p^2(z)\equiv\langle p^2\rangle=\dot{E}^2+\frac{I_0^2}{4E^2}=\left(\frac{d\sigma_x}{dz}\right)^2+\frac{I_0^2}{4\sigma_x{}^2}. \tag{18}$$

For the observable xp, the statistical average gives

$$\sigma_{xp}\equiv\langle xp\rangle=E\dot{E}=\frac{1}{2}\frac{d}{dz}\langle x^2\rangle=\frac{1}{2}\frac{d\sigma_x{}^2}{dz}, \tag{19}$$

and, finally, the mean value of the energy (4)

$$\mathcal{H}(z)\equiv\langle H\rangle=\frac{1}{2}\left(\dot{E}^2+\frac{I_0^2}{4E^2}\right)+\frac{1}{2}k_1E^2=\frac{\sigma_p^2}{2}+\frac{k_1(z)}{2}\sigma_x^2, \tag{20}$$

or, equivalently,

$$\mathcal{H}(z)=\frac{1}{2}\left(\frac{d\sigma_x}{dz}\right)^2+\frac{I_0^2}{8\sigma_x^2}+\frac{1}{2}k_1(z)\sigma_x^2. \tag{21}$$

Consequently, the Hamiltonian $\mathcal{H}$ defined by Eq. (21) has now the meaning of the averaged total energy associated with the transverse motion of the beam particles. It is very easy to prove the following very important relationships:

$$\frac{d^2\sigma_x^2}{dz^2}+4k_1(z)\sigma_x^2=4\mathcal{H} \tag{22}$$

and

$$\frac{d\mathcal{H}}{dz}=\left\langle\frac{\partial U}{\partial z}\right\rangle=\frac{1}{2}\dot{k}_1\sigma_x^2. \tag{23}$$

Remarkably, Eqs. (22) and (23) describe statistically (virial description) the behavior of the paraxial electronic rays in a linear lens of strength $k_1(z)$. But some additional information can be obtained from Eqs. (17)–(19). In fact, the quantities $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle xp\rangle$ are the elements of the *diffusion matrix* whose determinant essentially defines the square of the *diffusion coefficient*. Let us introduce the following quantity proportional to this coefficient and called *rms emittance* [41,42]:

$$\frac{\epsilon}{2}=[\langle x^2\rangle\langle p^2\rangle-\langle xp\rangle^2]^{1/2}. \tag{24}$$

Note that results (17)–(19) show us that both in the linear lens and in vacuo ϵ is an invariant and coincides with I_0:

$$\frac{I_0^2}{4}=\langle x^2\rangle\langle p^2\rangle-\langle xp\rangle^2. \tag{25}$$

For an arbitrary potential, ϵ is not necessarily preserved. Remarkably, from Eq. (24) in particular we have

$$\sigma_x\sigma_p\geqslant\frac{\epsilon}{2}. \tag{26}$$

We would like to stress that Eq. (14) represents a tautology, while the statistical form (26) actually represents a sort of *uncertainty relation* even if the particle beam is a *classical system*. Furthermore, it is clear that Eq. (26) defines the transverse beam emittance as the *minimum reachable uncertanty*. By using Eqs. (17)–(23), it is easy to see that this minimum is reached at the equilibrium condition ($d\sigma_x/dz=0$). At the equilibrium, the phase-space distribution for a sufficiently dilute beam is Gaussian in both configuration and momentum spaces. Let us take these two equilibrium distributions for the dimensionless Hamiltonian (4), namely, given by

$$n_p^{(0)}(p)=n_{p0}^{(0)}\exp\left[-\frac{p^2}{2\sigma_{p0}^2}\right], \tag{27}$$

where $\sigma_{p0}^2\equiv k_BT/(mc^2)\equiv\langle p^2\rangle_{z=0}$ (k_B and T being the Boltzmann constant and the transverse temperature of the system, respectively), and

$$n_x^{(0)}(x)=n_{x0}^{(0)}\exp\left[-\frac{x^2}{2\sigma_0^2}\right], \tag{28}$$

where $\sigma_0^2\equiv\langle x^2\rangle_{z=0}$. Note that $\langle xp\rangle=\langle xp\rangle_{z=0}=0$. Consequently, at equilibrium, Eq. (25) gives

$$\langle x^2\rangle_{z=0}^{1/2}\langle p^2\rangle_{z=0}^{1/2}=\frac{\epsilon}{2}, \tag{29}$$

which proves that the minimum product of the uncertainties is given at the equilibrium states and numerically coincides with half of the beam emittance, and now it is easy to prove that [33]

$$\frac{\epsilon}{2}=\left(\frac{k_BT}{mc^2}\right)^{1/2}\langle x^2\rangle_{z=0}^{1/2}=\frac{v_{\rm th}}{c}\sigma_0, \tag{30}$$

which shows explicitly the thermal nature of the beam emittance; $v_{\rm th}\equiv(k_BT/m)^{1/2}$ represents the transverse *thermal velocity* of the system. Consequently, ϵ scales as $\sqrt{T}$. The above results clearly show that, if the temperature of the system is not negligible, the electron rays are affected by a diffusion whose effect is to spread them out while the beam is propagating. This effect produces a *dispersion* among the electron rays in competition with tendency, due to the potential $U(x,z)$, to force them to be *ordered*. To see this diffusion effect more evidently, let us consider the special case of $U(x,z)=0$ (i.e., the beam is traveling in vacuo). In this case, Eqs. (21) and (23) imply that $\mathcal{H}$ is a positive constant given by

$$\mathcal{H}=\frac{1}{2}\left(\frac{d\sigma_x}{dz}\right)^2+\frac{\epsilon^2}{8\sigma_x^2}=\text{const}, \tag{31}$$

and, consequently, Eq. (22) becomes

$$\frac{d^2\sigma_x^2}{dz^2}=4\mathcal{H}=\text{const}, \tag{32}$$

which, for the intial condition $\sigma_x(z=0)\equiv\sigma_0$ and $\dot{\sigma}_x(z=0)=0$, gives

$$\sigma_x^2(z)=\sigma_0^2+2\mathcal{H}z^2=\sigma_0^2+\frac{\epsilon^2}{2\sigma_0^2}z^2. \tag{33}$$

This means that, while the beam is traveling from $z=-|\overline{z}|$, the electronic rays will not focus in a one point only. Starting from an initial spread $\overline{\sigma}\equiv\sigma_x(-|\overline{z}|)=\sigma_0[1+(\epsilon^2/2\sigma_0^4)\overline{z}^2]^{1/2}$, in the case of focusing the electron rays will reach the minimum spot σ_0 and then the beam will diverge, giving greater values of the spot. We want to point out that, since the particles are moving in vacuo, their trajectories must be straight. Even if the electron rays are straight, their mixing is due to the thermal spreading (diffusion) in such a way as to produce the beam envelope described by Eq. (33) (in 2 transverse space dimensions, it would be a hyperboloid of rotation around the z axis). The entity of this ray mixing is the order of $\epsilon^2/4\sigma_0^2\approx v_{\rm th}^2/c^2$.

For particle beams in the accelerators, typically $v_{\rm th}/c$ is much less than 1. In fact, transverse particle motion is classical while the longitudinal one is relativistic. So, the condition $v_{\rm th}/c\ll1$ is thus equivalent to consider in vacuo the envelope function $E(z)$ slowly varying with respect to the oscillating term $\cos\Delta\phi$. This is, in fact, consistent with the above paraxial approximation.

IV. PHASE-SPACE DESCRIPTION

The statistical description presented above allows us to understand that, for particle beams with finite emittance (temperature), the determination of an electronic ray at the arbitrary x position of the transverse plane given at each z is affected by an intrinsic uncertainty that cannot be reduced to zero. Only when the transverse temperature is exactly zero, the electronic ray mixing (diffusion) disappears and finding an electronic ray at a given transverse position is a deterministic operation based on simple geometrical arguments.

However, for finite-beam emittance, the intrinsic uncertainty on the transverse position at each z cannot allow for resolving among two or more rays in the sense that they are indistinguishable within this uncertainty which must be the order of $\sigma_x(z)$. In particular, at the focal point, it would be

σ_0. Consequently, for a finite emittance, we need to assign a probability (in principle, positive and finite) of finding an electronic ray at the transverse location x in the plane for given z. This probability distribution, say $P_x(x,z;\epsilon)$, would be both depending on the (transverse) emittance ϵ (i.e., transverse temperature) and normalized in the x space, namely,

$$\int_{-\infty}^{\infty} P_x(x,z;\epsilon)\,dx=1, \tag{34}$$

with the following physical meaning. Multiplying $P_x(x,z;\epsilon)$ by the total number of the beam particles, one obtains the transverse particle beam density (i.e., the electronic ray density with respect to the transverse direction).

In order to give the transverse beam dynamics description in terms of this probability distribution, let us start from the ordinary 2D phase-space equation for the electronic rays. To this end, we introduce the phase-space density distribution $\rho(x,p,z)$ in such a way to have for a generic observable $f(x,p)$ the following average:

$$\langle f(x,p)\rangle=\int f(x,p)\rho(x,p,z)\,dx\,dp, \tag{35}$$

provided that the following normalization condition holds

$$\int \rho(x,p,z)\,dx\,dp=1. \tag{36}$$

By definition ρ is constant of motion, and, consequently, must obey to the following phase-space equation [43]:

$$\frac{\partial\rho}{\partial z}+\{\rho,H\}=0, \tag{37}$$

where H is the Hamiltonian for an arbitrary potential given by Eq. (2). By using the Hamilton's equations, Eq. (37) can be explicitly written in the following form:

$$\frac{\partial\rho}{\partial z}+p\,\frac{\partial\rho}{\partial x}-\left(\frac{\partial U}{\partial x}\right)\frac{\partial\rho}{\partial p}=0, \tag{38}$$

which describes a phase-space evolution of electronic rays.

By introducing the dimensionless variables

$$\overline{z}\equiv\frac{z}{2\sigma_0},\qquad \overline{x}\equiv\frac{x}{2\sigma_0}, \tag{39}$$

Eq. (38) assumes the form

$$\frac{\partial\overline{\rho}}{\partial\overline{z}}+p\,\frac{\partial\overline{\rho}}{\partial\overline{x}}-\left(\frac{\partial\overline{U}}{\partial\overline{x}}\right)\frac{\partial\overline{\rho}}{\partial p}=0, \tag{40}$$

where $\overline{\rho}\equiv\rho(x/2\sigma_0,p,z/2\sigma_0)\equiv\overline{\rho}(\overline{x},p,\overline{z})$ and $\overline{U}\equiv U(x/2\sigma_0,z/2\sigma_0)\equiv\overline{U}(\overline{x},\overline{z})$.

However, we want to give a more interesting, but approximate *effective* electronic-ray description, taking explicitly into account their thermal spreading. According to the results of the previous section, since for finite emittance the indistinguishability among two or more rays due to the thermal spreading is the order of $\eta\equiv\epsilon/2\sigma_0=v_{\rm th}/c\ll 1$, $\partial\overline{U}/\partial\overline{x}$ in Eq. (40) can be conveniently replaced by the following symmetrized Schwarz-like finite difference ratio:

$$\frac{\partial\overline{U}}{\partial x}\approx\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{\eta}. \tag{41}$$

This way, Eq. (40) may be replaced by the following equation for an effective distribution, say $\overline{\rho_w}(\overline{x},p,\overline{z};\eta)$:

$$\frac{\partial\overline{\rho_w}}{\partial\overline{z}}+p\,\frac{\partial\overline{\rho_w}}{\partial\overline{x}}-\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{\eta}\,\frac{\partial\overline{\rho_w}}{\partial p}=0. \tag{42}$$

The *transition* from Eqs. (40) to (42), based on physical arguments, is partially a change of partial differential equation [i.e., Eq. (40)] to differential-difference equation [i.e., Eq. (42)], which may be considered as ansatz of a *deformation* of the 2D phase-space equation for the electronic rays.

Given the smallness of η, multiplying both numerator and denominator of the last term of the left-hand side by the imaginary unit i, we have

$$\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{i\eta}\,i\,\frac{\partial\overline{\rho_w}}{\partial p}\approx\frac{\overline{U}(\overline{x}+(i\eta/2)\partial/\partial p)-\overline{U}(\overline{x}-(i\eta/2)\partial/\partial p)}{i\eta}\overline{\rho_w}. \tag{43}$$

Thus, going back to the old variables x and z, Eq. (42) assumes formally the look of a von Neumann equation [18,44] (let us say von Neumann–like equation):

$$\left\{\frac{\partial}{\partial z}+p\,\frac{\partial}{\partial x}+\frac{i}{\epsilon}\left[U\left(x+i\frac{\epsilon}{2}\frac{\partial}{\partial p}\right)-U\left(x-i\frac{\epsilon}{2}\frac{\partial}{\partial p}\right)\right]\right\}\rho_w=0, \tag{44}$$

where $\rho_w\equiv\overline{\rho_w}(2\sigma_0\overline{x},p,2\sigma_0\overline{z};2\sigma_0\eta)\equiv\rho_w(x,p,z;\epsilon)$. Equation (44) shows that, in the framework of this effective description, the phase-space evolution equation for electronic rays is a quantumlike phase-space equation where $\hbar$ and the time t are replaced by the emittance ϵ and the propagation coordinate z, respectively.

However, some considerations are in order. (i) Approximation (41) is due both to the smallness of η and the fact that evaluation of $\overline{U}$ variation around the location $\overline{x}$ does not make sense within an interval of size η. This, in fact, corresponds to the intrinsic uncertainty produced among the rays by the finite-temperature spreading. In other words, thermal mixing of electronic rays affects the evaluation of U variation with respect to x. Thus, Eq. (42) represents a possible way to take into account the ray mixing in this evaluation.

(ii) Since $\overline{U}(\overline{x}+(i\eta/2)\partial/\partial p)-\overline{U}(\overline{x}-(i\eta/2)\partial/\partial p)=(\partial\overline{U}/\partial\overline{x})i\eta(\partial/\partial p)+O(\eta^3\partial^3/\partial p^3)$, approximation (43) is equivalent to assume that terms $O(\eta^3\partial^3/\partial p^3)$ are small corrections compared to the lower-order ones, according to the paraxial approximation. Consequently, from the quantumlike point of view, approximation (43) plays a role analogous to the one played by the *semiclassical approximation* [45].

(iii) While the distribution $\rho(x,p,z)$ involved in Eq. (38) is introduced in a classical framework and it is positive definite, the function $\rho_w(x,p,z;\epsilon)$ is introduced in a quantumlike framework, which plays the role of an *effective* description taking into account the thermal spreading among the electronic rays. In addition, in this context $\rho_w(x,p,z;\epsilon)$ cannot be used to give information within the phase-space cells with size smaller than ϵ, due to the intrinsic uncertainty exhibited by the system for finite temperatures, i.e., due to the indistinguishability among the electronic rays. Consequently, we would expect that ρ_w violates the positivity definiteness within some phase-space regions. On the other hand, even with the limitations given by points (i) and (ii), it is clear from the von Neumann–like equation (44) that ρ_w is a sort of *Wigner-like* function. Thus, it is not positive definite, due to the quantumlike uncertainty principle given in Sec. III. This means that, in analogy with quantum mechanics, $\rho_w(x,p,z;\epsilon)$ can be defined as *quasidistribution* even its x projection and p projection are actually configuration-space distribution and momentum-space distribution, respectively. In particular, within the framework of the above effective description of the electronic ray evolution, we assume that the probability $P_x(x,z;\epsilon)$ introduced above is

$$P_x(x,z;\epsilon)=\int \rho_w(x,p,z;\epsilon)\,dp, \tag{45}$$

provided that also ρ_w is normalized over the phase space.

Note that, for arbitrary U:

$$\lim_{\epsilon\to 0}\rho_w(x,p,z;\epsilon)=P^{(0)}(x,z)\,\delta(p-V^{(0)}(x,z))\equiv\rho_0(x,p,z), \tag{46}$$

which describes the (transverse) phase-space motion of a *cold* beam. Multiplying the total number of particles by $P^{(0)}(x,z)$ we obtain the transverse space density of the electronic rays at each z for a cold beam. Furthermore, $V^{(0)}(x,z)$ is the (transverse) current velocity, which in this case obeys, with $P^{(0)}(x,z)$, the following equations:

$$\frac{\partial P^{(0)}}{\partial z}+\frac{\partial}{\partial x}(P^{(0)}V^{(0)})=0 \tag{47}$$

(continuity equation),

$$\left(\frac{\partial}{\partial z}+V^{(0)}\frac{\partial}{\partial x}\right)V^{(0)}=-\frac{\partial U}{\partial x} \tag{48}$$

(fluid motion equation). Note that in the above limit the local slope of the electronic rays $p=dx/dz$ is determined only by the gradient of U. In particular, in vacuo $(U=0)$ a cold uniform beam has phase-space density of the form $P_0\delta(p-V_0)$, with P_0 and V_0 constants. With the language of particle accelerator physics, this kind of beam is called *monochromatic beam*. It is easy to see that all the electron rays of a monochromatic beam have the same slope.

The above results allow us to write that, for an arbitrary potential, we have

$$\lim_{\epsilon\to 0}P_x(x,z;\epsilon)=P^{(0)}(x,z). \tag{49}$$

Remarkably, from the above results it follows that it may exist a complex function, say $\Psi(x,z)$, such that

$$P_x(x,z;\epsilon)=\Psi(x,z)\Psi^*(x,z), \tag{50}$$

used also for a description of pure quantum states, and the following *quantumlike density matrix*

$$G(x,x',z)=\Psi(x,z)\Psi^*(x',z), \tag{51}$$

used also for description of mixed quantum states, connected with ρ_w by means of the following Wigner-like transformation:

$$\rho_w(x,p,z;\epsilon)=\frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}G\left(x+\frac{y}{2},x-\frac{y}{2},z\right)\exp\left(i\frac{py}{\epsilon}\right)dy, \tag{52}$$

or, for pure states,

$$\rho_w(x,p,z;\epsilon)=\frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}\Psi^*\left(x+\frac{y}{2},z\right)\times\Psi\left(x-\frac{y}{2},z\right)\exp\left(i\frac{py}{\epsilon}\right)dy. \tag{53}$$

Consequently, $\Psi(x,z)$ must obey to the following Schrödinger-like equation:

$$i\epsilon\frac{\partial\Psi}{\partial z}=-\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\Psi+U(x,z)\Psi. \tag{54}$$

This equation has been the starting point to construct the quantumlike approach of charged-particle beams, which is known in the literature as the thermal wave model (TWM). It has been applied to a number of problems in particle accelerators and plasma physics [19–25]. TWM assumes that the transverse (longitudinal) dynamics of a charged particle beam, interacting with the surroundings, is governed by a Schrödinger-like equation for a complex function in which Planck's constant is replaced by the transverse (longitudinal) beam emittance. This complex function, called beam wave function (BWF) has the following meaning: its squared modulus is proportional to the transverse (longitudinal) beam density. This way the beam as a whole is thought of as a single quantumlike particle whose *diffractionlike* spreading due to the emittance accounts for the *thermal spreading*.

V. QUANTUMLIKE CORRECTIONS

In this section, we analyze the quantumlike corrections [46] that the *effective* electronic ray description presented above gives with respect to the *pure* classical treatment. In other words, we make a comparison between the quantumlike description given by Eq. (44) and the one given by Eq. (38). To this end, we first observe that, if the beam is in a quadrupole (linear lens), Eq. (44) collapses in Eq. (38) and no quantumlike corrections are present. One can calculate the set of moment equations associated with Eqs. (38) and (44), respectively. Defining the following *Liouville operator*

$$\hat{\mathcal{L}} \equiv \frac{\partial}{\partial z} + p\frac{\partial}{\partial x} - \left(\frac{\partial U}{\partial x}\right)\frac{\partial}{\partial p}, \tag{55}$$

where U is an arbitrary potential that can be expanded in Taylor series with respect to x, it is easy to see that Eqs. (38) and (44) become, respectively,

$$\hat{\mathcal{L}}\rho_w = 0, \tag{56}$$

and

$$\hat{\mathcal{L}}\rho_w = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\epsilon}{2}\right)^{2k} \frac{\partial^{2k+1}U}{\partial x^{2k+1}} \frac{\partial^{2k+1}\rho_w}{\partial p^{2k+1}}, \tag{57}$$

(each k being a non-negative integer). Note that the right-hand side of Eq. (57) contains an extraterm that is not present in the classical form (56). By introducing the ν-order (ν being a non-negative integer) moment of $\hat{\mathcal{L}}$ as

$$\mathcal{M}^{(\nu)}(x,z) \equiv \int_{-\infty}^{\infty} p^{\nu} \hat{\mathcal{L}}\rho_w dp, \tag{58}$$

the classical equation (56) leads to

$$\mathcal{M}^{(\nu)}(x,z) = 0, \quad \forall \nu \geqslant 0, \tag{59}$$

which, in turn, gives the continuity equation

$$\frac{\partial P_x}{\partial z} + \frac{\partial}{\partial x}(P_x V) = 0 \tag{60}$$

for $\nu=0$, the motion equation

$$\left(\frac{\partial}{\partial z} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} - \frac{1}{P_x}\frac{\partial \Pi}{\partial x}, \tag{61}$$

for $\nu=1$, the energy equation

$$\mathcal{M}^{(2)}(x,z) = 0, \tag{62}$$

for $\nu=2$, and so on, where

$$V(x,z) = \frac{1}{P_x}\int_{-\infty}^{\infty} p\rho_w dp \tag{63}$$

is the current velocity, which is experimentally the first-order moment of ρ_w, and

$$\Pi(x,z) \equiv \int_{-\infty}^{\infty} (p-V)^2 \rho_w dp \tag{64}$$

is the kinetic pressure (divided by the total number of the particles) or the second-order moment of ρ_w.

On the other hand, the quantumlike equation (57) gives

$$\mathcal{M}^{(\nu)}(x,z) = 0, \quad \nu = 0,1,2 \tag{65}$$

and

$$\begin{aligned}\mathcal{M}^{(\nu)}&(x,z)\\ &= -\sum_{k=1}^{k_{\max}\leqslant(\nu-1)/2} (-1)^k \binom{\nu}{2k+1}\\ &\quad\times\left(\frac{\epsilon}{2}\right)^{2k} \frac{\partial^{2k+1}U}{\partial x^{2k+1}} \int_{-\infty}^{\infty} p^{\nu-2k-1}\rho_w dp\\ &\neq 0 \quad \forall \quad \nu \geqslant 3.\end{aligned} \tag{66}$$

Consequently, for an arbitrary potential and up to the energy equations, the two descriptions (the classical and the quantumlike) coincide. The discrepancy appears at the order equal to or higher than the third one in the moment equations. In principle, Eqs. (56) and (57) are equivalent to an infinite set of their moment equations (59) and (65)–(66), respectively. The characteristic of these moment equations is that the one of ν order is an evolution for the ν-order moment of ρ_w, but contains a $(\nu+1)$-order moment of this function. Provided that a *closure equation* is introduced, which relates a $(\nu+1)$-order moment with the lower-order ones, the truncated set of equations, consisting of moment equations up to the ν order plus the closure equation, is fully equivalent to Eqs. (56) or (57), respectively.

Usually, the lowest order of truncation is introduced for $\nu=1$, by introducing, for the transverse dynamics, the following ideal gas state equation [46] (isothermal approximation):

$$P_x \frac{k_B T}{mc^2} = \Pi. \tag{67}$$

In fact, even if the beam propagates along z with relativistic motion, the transverse particle motion around z is classical. Consequently, the beam behaves transversally like a nonrelativistic ideal gas. Moreover, note that, denoting with N the total number of beam particles, the quantity $n \equiv NP_x$ is the transverse number density of the beam. At this level, we are describing our beam in terms of the fluid theory

$$\frac{\partial P_x}{\partial z} + \frac{\partial}{\partial x}(P_x V) = 0, \tag{68}$$

$$\left(\frac{\partial}{\partial z} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} - \frac{v_{\text{th}}^2}{c^2}\frac{1}{P_x}\frac{\partial P_x}{\partial x}. \tag{69}$$

It is obvious from Eqs. (59)–(66) that *the classical and the quantumlike descriptions coincide at the level of the fluid theory for noncold beams*. Note that, in particular, in the limit $\epsilon \to 0$, Eqs. (68) and (69) recover Eqs. (47) and (48), respectively.

Going on to $\nu=2$, for the truncation a closure equation involving the moments of third order and the lower ones have to be introduced. By virtue of Eqs. (60)–(62) and (65), *the descriptions coincide also at this level if a suitable closure equation is chosen for both*.

For orders $\nu \geqslant 3$, according to Eq. (66), the truncation cannot allow for having the equivalence between the classical and the quantumlike descriptions. In particular, the third-order moment equation ($\nu=3$) of Eq. (57) is

$$\mathcal{M}^{(3)}=\left(\frac{\epsilon}{2}\right)^2\left(\frac{\partial^3 U}{\partial x^3}\right)P_x, \tag{70}$$

and the one for the fourth-order moment ($\nu=4$) is

$$\mathcal{M}^{(4)}=4\left(\frac{\epsilon}{2}\right)^2\left(\frac{\partial^3 U}{\partial x^3}\right)P_x V. \tag{71}$$

The above analysis allows us to conclude that *at the third-order moment description the discrepancy between the classical and the quantumlike descriptions appears as a delicate effect*. In fact, for arbitrary potentials and for given emittances, the discrepancy increases as the density of the beam. Thus, to make it evident, very intense beams are necessary.

In addition, if U is a symmetric potential with respect to the propagation direction z, i.e., $U(-x,z)=U(x,z)$, the discrepancy corresponding to the third and the fourth-order moments are still negligible for beams that are mainly concentrated around z (x very close to zero, i.e., paraxial beam), because in this case $\partial^3 U/\partial x^3 \propto x \approx 0$.

VI. CONCLUSIONS, REMARKS, AND FUTURE PERSPECTIVES

In this paper, the charged-particle beam transport has been investigated with a quantum-like approach. By starting from the electronic-ray concept in paraxial approximation, we have given the statistical description of the electronic ray evolution, which has allowed us to obtain a quantumlike picture of a charged-particle beam transport, where a sort of quantumlike uncertainty principle holds for the spread of particle position distribution and the spread of particle momentum. This way we first introduced a sort of *Wigner-like* picture behind the electronic ray evolution and then recovered the already known quantumlike description for charged-particle beam dynamics called thermal wave model [19–25]. Within the framework of the Wigner-like picture, the quantumlike corrections have been introduced and compared with the standard classical picture for arbitrary potentials, showing that the above quantumlike approach could be a useful tool for particle accelerator physics investigation. It is worth mentioning that this comparison is in agreement with a recent numerical phase-space analysis that compares the quantumlike Wigner function of a charged-particle beam in a quadrupole with small sextupole and octupole aberrations with the results of a standard particle tracking code simulation [25].

However, the following question naturally arises: *What would be the precise relation between the new Wigner-like formalism and the previous thermal wave model?* Well, TMW provides for a quantumlike description of charged-particle beam transport. This way, using the formal apparatus of quantum mechanics, it is possible to introduce the Wigner transform that connects the description in configuration space (in terms of BWF) with the one in the phase space (in terms of a Wigner function). In the early formulation of TWM [19–25], the above quantumlike (in particular, the above Wigner-like) picture was assumed valid beyond the semiclassical approximation. On the other hand, in the present paper, the *transition* from the classical phase-space equation to the *deformed* phase-space equation is valid *only* in the semiclassical approximation. Consequently, the *deformation method* presented in this paper allows us to recover the quantumlike picture, and in particular the Wigner-like picture, only in the semiclassical approximation. Furthermore, the analysis of the quantumlike corrections shows also that the above fluid description charged-particle beam transport can be thought of in terms of a semiclassical approximation of the moment hierarchy description. In fact, at the fluid level, the truncation is made at order $v_{\mathrm{th}}^2/c^2 \propto \epsilon^2$.

Nevertheless, we want to remark that this approach could be relevant also for a wide spectrum of topics in em radiation optics, general quantum mechanics, and quantum optics for the considerations that are in order.

(1) Equation (44) collapses in Eq. (38) in the case of a quadrupole (harmonic oscillator). However, due to the Wigner-like picture, Eq. (44) describes some states that are not described by its classical counterpart. In other words, the similarity between ρ and ρ_w in the harmonic oscillator is not possible for all the states. This makes evident a *quantumlike effect* that ρ_w contains and that ρ does not contain. Equations (40) and (42) become the same equation in the case of a quadrupole, where ϵ does not appear explicitly. However, a possible normalized solution of this phase-space equation for harmonic oscillator is [25]

$$\frac{1}{\pi\epsilon}\exp\left[-\frac{2}{\epsilon}[\gamma(z)x^2+2\alpha(z)xp+\beta(z)p^2]\right], \tag{72}$$

which explictly depends on ϵ. Consequently, in principle, to recover classical solutions we do not need necessarily to take the limit $\epsilon\rightarrow 0$. In this limit, we can recover the special family of classical solutions that describe the cold-beam transport only, as pointed out in Sec. IV. This means that Eq. (38) contains something more than the classical limit. In fact, solution (72), in which ϵ is a finite quantity, leads easily to the quantumlike uncertainty relation (26).

(2) From Eq. (57) it is clear that for finite emittance but in the case in which $(\epsilon/2)^2\partial^2\rho_w/\partial p^2 \gg (\epsilon/2)^s\partial^s\rho_w/\partial p^s$ for $s\geq 3$, Eqs. (44) and (38) formally coincide for an arbitrary (anharmonic) potential. However, also in this case, ρ_w contains, in principle, the quantumlike effects that ρ does not contain. Of course, according to the investigation about the discrepancy given in Sec. V, *this quantumlike effect is delicate for dilute beams but not, in principle, negligible*.

Thus, we can conclude that, for finite temperature: (a) in vacuo as well as for harmonic potentials, the deformed equation appears formally indistinguishable from its classical counterpart (also beyond the semiclassical approximation), but the former admits a wider class of solutions, which can be also negative; (b) in the case of anharmonic potentials, the deformed equation represents an *effective version* of its classical counterpart (on phase-space scale greater than ϵ). However, it coincides with the von Neumann equation in semiclassical approximation only. Consequently, also for these potentials it admits a solution that can be negative.

At this point, another natural question arises: *Do the new nonclassical states predicted by the Wigner picture contain physically relevant information*? In the quantumlike framework, these solutions describe excited states of the beam transport that are not considered in the classical picture. In reality, it seems that they could be the quantumlike version of the beam states corresponding to energies greater than the one associated with the lowest energy (fundamental state), which corresponds to a positive distribution with all the classical probability features. However, the above excited states can be negative due to the quantumlike uncertainty relation introduced by the indistinguishability among the electronic rays, but it seems clear that they are intrinsically classical states that cannot give suitable information within phase-space cells with size of order ϵ. And this is only produced in order to take into account the above loss of information within these cells, by the deformation of the classical phase-space equation.

A third question could be formulated as follows: *Does the above quantumlike formalism provide any new physical insight into the beam dynamics or it is just a different but nevertheless more complicated way of expressing ''familiar physics''*? Since we are not allowed, for a given finite temperature, to locate exactly an electronic ray within phase-space cells of order ϵ, the quantumlike description naturally replaces the classical one. Consequently, the above states associated with negative phase-space distribution can be considered as nonclassical (in the sense of quantumlike) states. Of course, by reducing the temperature more and more, all the classsical features are more and more recovered in such a way that the *nonclassicality* disappears. This aspect surely represents a new insight with respect to the ordinary classical description of charged-particle beam transport, but keeping in mind that the above quantumlike approach is capable of describing a wealth of phenomena, taking into account the thermal noise involved in the particle beam transport in a way (even effective) that the pure classical approach cannot give. Consequently, it is clear that the above quantumlike approach is not a more complicated way of expressing familiar physics. In fact, the quantumlike formalism allows us to solve problems of particle beam transport by using all the knowledge acquired thanks to very well tested techniques of solving quantum mechanical problems during a period longer than half a century.

Finally, we want to remark that, even if we have given a quantumlike picture for charged-particle beam transport, fully consistent with the quantumlike uncertainty relation (26), our description does not contradict classical mechanics. In fact, while $\hbar$ is a fundamental, universal constant, ϵ does not have such properties. Since the latter depends on the thermal noise, we can, *in principle*, arrange a series of experimental devices in which the temperature is progressively reduced. This way, we enhance the accuracy in finding the electronic ray location by reducing the thermal uncertainty more and more. Consequently, the quantumlike uncertainty *in principle* collapses into the *classical* independence between measuring of spot size and momentum spread. In this sense, our *effective* description is formally *quantumlike* but intrinsically *classical*. Of course, a *natural* limitation in reducing the thermal noise is established by the *proper* quantum uncertainty relation, which states that quantum fluctuations are *unavoidable* and *intrinsic*. In fact, the nature of the physical systems is basically *quantum* and not *classical*, but *this is true for all the systems in nature and not only for charged-particle beams*.

We observe that in quantum mechanics and in quantum optics measuring of the states described by Wigner functions was recently reduced by means of tomographic procedures to measure a positive marginal distribution related to the Wigner function by an integral transform (the Radon transform of optical tomography method [47,48] or Fourier transform of symplectic tomography [27,28]). Thus, we could state that, analogously, in the above quantumlike approach there is a possibility to transit from the classical phase-space equation to an equation for a positive marginal distribution of two types [28], which has standard classical features. This very important problem is considered in a forthcoming paper.

ACKNOWLEDGMENTS

The authors are very grateful to Professor Margarita A. Man'ko for reading the manuscript and giving useful suggestions.

[1] *Quantum-like Models and Coherent Effects*, edited by R. Fedele and P. K. Shukla (World Scientific, Singapore, 1995).
[2] D. Gloge and D. Marcuse, J. Opt. Soc. Am. **59**, 1629 (1969).
[3] M. Leontovich and V. Fock, Zh. Eksp. Teor. Fiz **16**, 557 (1946); M. A. Leontovich, Izv. Akad. Nauk SSSR **8**, 16 (1944).
[4] S. A. Ackhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Sov. Phys. Usp. **10**, 609 (1968).
[5] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley-Interscience Publication, New York, 1984).
[6] V. I. Karpman, *Non-linear Waves in Dispersive Media* (Pergamon Press, Oxford, 1975).
[7] D. Marcuse, *Light Transmission Optics* (Van Nostrand, New York, 1972).
[8] J. A. Arnoud, *Beam and Fiber Optics* (Academic Press, New York, 1976).
[9] R. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
[10] J. R. Klauder, J. Math. Phys. **5**, 177 (1964).
[11] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[12] *Proceedings of the Second International Workshop on Squeezed States and Uncertainty Relations, Moscow, 25–29 May 1992*, NASA Conference Publication No. 3219 edited by D. Han, Y. S. Kim, and V. I. Man'ko (National Aeronautics and Space Administration, Goddard Space Flight Center, Greenbelt, MD, 1993).
[13] J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).
[14] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[15] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[16] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica (Amsterdam) **72**, 597 (1974).
[17] A. L. Rivera, N. M. Atakishiyev, S. M. Chumakov, and K. B. Wolf, Phys. Rev. A **55**, 876 (1997).
[18] E. Wigner, Phys. Rev. **40**, 749 (1932).
[19] R. Fedele and G. Miele, Nuovo Cimento D **13**, 1527 (1991).

[20] R. Fedele and G. Miele, Phys. Rev. A **46**, 6634 (1992).
[21] R. Fedele and P. K. Shukla, Phys. Rev. A **44**, 4045 (1992).
[22] S. De Nicola, R. Fedele, V. I. Man'ko, and G. Miele, Phys. Scr. **52**, 191 (1995).
[23] R. Fedele, G. Miele, L. Palumbo, and V. G. Vaccaro, Phys. Lett. A **179**, 407 (1993).
[24] R. Fedele, F. Galluccio, and G. Miele, Phys. Lett. A **185**, 93 (1994).
[25] R. Fedele, F. Galluccio, V. I. Man'ko, and G. Miele, Phys. Lett. A **209**, 263 (1995).
[26] J. Lawson, P. M. Lapostolle, and R. L. Gluckstern, Part. Accel. **5**, 61 (1973).
[27] S. Mancini, V. I. Man'ko, and P. Tombesi, Quantum Semiclassic. Opt. **7**, 615 (1995).
[28] G. M. D'Ariano, S. Mancini, V. I. Man'ko, and P. Tombesi, Quantum Semiclassic. Opt. **8**, 1017 (1996).
[29] S. Mancini, V. I. Man'ko, and P. Tombesi, Phys. Lett. A **213**, 1 (1996).
[30] P. Sturrock, *Static and Dynamic Electron Optics* (Cambridge University Press, London, 1955).
[31] W. Glaser, *Grundlangen der Elektroneoptik* (Springer-Verlag, Vienna, 1952); O. Klemperer and M.E. Barnett, *Electron Optics*, 3rd ed. (Cambridge University Press, London, 1955); V. K. Zworykin, G. A. Morton, E. G. Ramberg, J. Hillier, and A. W. Vance, *Electron Optics and Electron Microscope* (John Wiley, New York, 1945).
[32] L. V. Tarasov, *Physical Principle of Quantum Electronics (Optical Region)* (Sov. Radio, Moscow, 1976); D. N. Nikogosyan, Kvant. Elektron. (Moscow) **4**, 1 (1977); **4**, 5 (1977).
[33] J. Lawson, *The Physics of Charged-Particle Beams*, 2nd ed. (Clarendon Press, Oxford, 1988).
[34] M. S. Livingstone and J. P. Blewett, *Particle Accelerators* (McGraw-Hill, New York, 1962).
[35] K. Steffen, in *Proceedings of CERN Accelerator School, Gif-sur-Yvette, Paris, September 3-14 1984*, edited by P. Briant and S. Turner (CERN, Geneva, 1985), Vol. 1, p. 25.
[36] L. D. Landau and E. M. Lifshits, *Mechanics* (Pergamon Press, Oxford, 1957).
[37] P. Ermakov, Univ. Izv. (Kiev), **20**, 1 (1880).
[38] H. R. Lewis, Phys. Rev. Lett. **18**, 510 (1967); **18**, 636 (1967).
[39] H. R. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[40] M. Sands, in *The Physics of Electron Storage Rings. An Introduction*, in *Rendiconti della Scuola Internazionale di Fisica, ''Enrico Fermi,'' Course XLVI*, edited by B. Toushek (Academic Press, New York, 1971).
[41] P. M. Lapostolle, IEEE Trans. Nucl. Sci. **18**, 1101 (1971).
[42] F. J. Sacherer, IEEE Trans. Nucl. Sci. **18**, 1105 (1971).
[43] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, Oxford, 1957).
[44] M. Toda, R. Kubo, and N. Saitô, *Statistical Physics I*, 2nd ed. (Springer-Verlag, Berlin, 1995), p. 24; M. Hillery, R. F. O'Connell, M. D. Scully, and E. Wigner, Phys. Rep. **106**, 121 (1984).
[45] E. J. Heller, J. Chem. Phys. **67**, 3339 (1977).
[46] R. Fedele and G. Miele, in *New Perspectives in the Physics of Mesoscopic Systems*, edited by S. De Martino *et al.* (World Scientific, Singapore, 1997), p. 120.
[47] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
[48] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993).